

GLOBAL MATRIX FACTORIZATIONS

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ABSTRACT. We study matrix factorization and curved module categories for Landau–Ginzburg models (X, W) with X a smooth quasi-projective variety, extending parts of the work of Dyckerhoff for the case of affine X . We equip these categories with model category structures, extending the work of Positselski. Using results of Rouquier and Orlov, we obtain compact generators for our categories. Via Toën’s derived Morita theory, we identify Hochschild cohomology with derived endomorphisms of the diagonal curved module. We compute the latter and get the expected result. Finally, we show that our categories are smooth, proper when the singular locus of W is proper, and Calabi–Yau when X is Calabi–Yau.

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1. INTRODUCTION

Recall the prototypical statement of Homological Mirror Symmetry [Kon]: For every Calabi–Yau manifold Y , there is a mirror Calabi–Yau manifold X such that the Fukaya category [FOOO] (resp. derived category) of Y is equivalent to the derived category (resp. Fukaya category) of X . Suppose now that Y is not a Calabi–Yau manifold but, say, a smooth toric Fano variety considered as a symplectic manifold. Then the mirror of X is expected to be a *complex Landau–Ginzburg model* [Aur, Cla]: a pair (X, W) , where X is a complex manifold or variety and W is a holomorphic or regular function. The function W is called the *superpotential*. The statement of Homological Mirror Symmetry becomes: The Fukaya category of Y is equivalent to the *matrix factorization category* [KapLi] of (X, W) .¹

If $X = \text{Spec } A$ for a commutative finite type \mathbb{C} -algebra A and if $W \in A$ has a single critical value, which we assume to be $0 \in \mathbb{C}$, then the differential $\mathbb{Z}/2\mathbb{Z}$ -graded category of matrix factorizations $\text{MF}(X, W)$ is defined as follows. This category has as objects

$$P = (P_1 \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{p_0} \end{array} P_0)$$

¹Moreover when Y has a complex structure and (X, W) has a symplectic structure, then the derived category of Y should be equivalent to the Fukaya–Seidel category [Sei] of (X, W) .

where the P_i are finitely generated projective A -modules, and the p_i are A -module morphisms satisfying $p_{i+1} \circ p_i = W \cdot \text{id}_{P_i}$.² For the morphisms between P and P' , one takes the $\mathbb{Z}/2\mathbb{Z}$ -graded complex of all A -module morphisms

$$\text{Hom}(P, P') = \bigoplus_{i,j} \text{Hom}_A(P_i, P'_j)$$

with grading given by $i + j$ (modulo 2), and with the differential

$$\partial : f \longmapsto p' \circ f - (-1)^{|f|} f \circ p$$

for homogeneous f . For more details, see §3 of [Orl1]. We may also refer to objects of this category as *curved $\mathbb{Z}/2\mathbb{Z}$ -graded complexes of finitely generated projective A -modules with curvature W* [Pos]. We may truncate this in various ways: *curved complexes of projective modules*, *curved projective modules*, etc. More generally, we will consider *curved modules* of various kinds, not necessarily finitely generated or projective. We also have the functor $P \mapsto P^\#$ sending a curved object to the underlying $\mathbb{Z}/2\mathbb{Z}$ -graded object gotten by forgetting p_0 and p_1 .

When A is moreover regular and local, and if W has an isolated singularity at the unique closed point of $\text{Spec } A$, Dyckerhoff [Dyc, DycMur] has proven that $\text{MF}(X, W)$ is a smooth and proper Calabi–Yau category satisfying the Hodge-to-de Rham (Hochschild-to-periodic cyclic) degeneration, and thus it gives rise to a 2D TQFT that extends to the Deligne–Mumford boundary [KonSoi, Cos, Lur, KatKonPan, PolVai].

In this paper, we extend the theory of matrix factorizations to the case of Landau–Ginzburg models (X, W) where X is not necessarily affine. So let X be a smooth quasi-projective \mathbb{C} -scheme, and let W be a regular function which defines a flat map $X \rightarrow \mathbb{A}_{\mathbb{C}}^1$. Replacing A -modules with sheaves of \mathcal{O}_X -modules, the above definition of matrix factorizations still makes sense — to be precise, matrix factorizations are now defined to be curved complexes of vector bundles, i.e. locally free sheaves of finite type. However, as is briefly discussed in [KatKonPan], the “correct” definition of the matrix factorization category in the non-affine situation should take into account the nonvanishing of higher sheaf cohomology. This means that roughly speaking, we should replace the complex $\text{Hom}(P, P')$ with some form of a derived complex $\mathbb{R}\text{Hom}(P, P')$, for instance via a Čech or Dolbeault resolution of the sheaf $\mathcal{H}\text{om}_{\mathcal{O}_X}(P, P')$.

To make this precise, we consider in section 2 of this paper the category $\text{QCoh}(X, W)$ of curved complexes of quasi-coherent sheaves. Following Positselski’s lead in the case of affine X [Pos], we equip this category with a model category structure for which fibrant objects are curved complexes of injective sheaves. This gives rise to the dg category $\text{Inj}(X, W)$, which is a dg enhancement of the absolute derived category $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$. Via fibrant replacement, we define the derived complex $\mathbb{R}\text{Hom}(P, P')$ of morphisms and thusly the “correct” matrix factorization dg category $\text{MF}_{\text{dg}}(X, W)$. Furthermore, again following Positselski, we show that matrix factorizations are compact as objects of $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$ and that the idempotent completion of the subcategory thereof recovers $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)_c$, where the subscript “c” denotes the subcategory of all compact objects.

In section 3, we compute the Hochschild cohomology of $\text{Inj}(X, W)$ and hence that of $\text{MF}_{\text{dg}}(X, W)$, yielding a result which was anticipated in [KatKonPan]. The approach is similar to the one taken in [Dyc] for the affine situation — we find a compact generator of the category, and it follows that the category is equivalent to the dg derived category of the endomorphism dg algebra of the generator. This in turn allows us to apply the derived Morita theory of [Toë]. To reach our Hochschild cohomology result, we take a detour into the work of [Yek] on the global Hochschild–Kostant–Rosenberg theorem, and we employ the

²The nomenclature “matrix factorization” is due to Eisenbud [Eis] and comes from the fact that when the P_i are free modules, the d_i can be thought of as matrices with entries in A that factorize the scalar matrices $W \cdot \text{id}_{P_i}$.

calculations of [CalTu]. It also follows from results in this section that $\mathrm{MF}_{\mathrm{dg}}(X, W)$ is smooth, and that it is proper when the singular locus of W is proper.

Finally, in section 4, using some standard results from [HarRD], we show that if the Landau-Ginzburg model (X, W) satisfies the condition that X is Calabi-Yau, then $\mathrm{MF}_{\mathrm{dg}}(X, W)$ is a Calabi-Yau category. We remark that our proof of the Calabi-Yau condition on the category mimics Dyckerhoff's proof, except that we are able to identify explicitly how and where the Calabi-Yau condition on the space X comes into play. This is not immediately transparent in Dyckerhoff's proof, since in his local situation the Calabi-Yau condition on X is automatic.

In comparison with earlier works, the new features in this work are the systematic use of Positselski's theory and the identification of generators. Dyckerhoff identifies generators for matrix factorization categories in the local situation, but we identify generators in the global situation for the corresponding derived categories of singularities of the zero fibers using the work of Rouquier [Rou]. A theorem of Orlov [Orl2] says that the two are the same. On the other hand, Dyckerhoff is able to explicitly compute the endomorphism dg algebras of his generators. In our case, there is no clear way in general to associate an explicit matrix factorization to a generator of the derived category of singularities, and no clear way to compute the endomorphism dg algebra. We hope to return to this computational problem in the future.

For ease of notation and exposition, we always assume that our superpotentials W have a single critical value $0 \in \mathbb{C}$. If there are multiple critical values c_i , then the results all generalize trivially by considering the product $\prod_i \mathrm{MF}(X, W - c_i)$ instead of $\mathrm{MF}(X, W)$, etc. Furthermore, unless specified otherwise, when we say *dg category* we will always mean differential $\mathbb{Z}/2\mathbb{Z}$ -graded category, that is, a category enriched over the category of $\mathbb{Z}/2\mathbb{Z}$ -graded complexes of \mathbb{C} -vector spaces. More generally, all of our graded objects are $\mathbb{Z}/2\mathbb{Z}$ -graded objects unless specified otherwise.

It is probably more standard for “dg” to mean differential \mathbb{Z} -graded, and indeed the categories arising in Mirror Symmetry are often so. It is in fact possible to define a notion of \mathbb{Z} -graded matrix factorization and corresponding differential \mathbb{Z} -graded categories thereof for which results similar to those in this paper hold. This will not be addressed in this paper. However, as this work was in progress, we were informed that Anatoly Preygel has done work in this direction using an exciting different approach. Our present paper builds heavily on ideas and methods of Dyckerhoff, Orlov, Positselski, Rouquier, and Toën; on the other hand Preygel's paper is inspired by an idea of Constantin Teleman and Jacob Lurie, and uses the methods of Lurie's derived algebraic geometry.

In future work, we hope to explore explicit examples of the general theory developed here, particularly examples of relevance to Mirror Symmetry. In addition, we hope to extend the theory of matrix factorizations for Landau-Ginzburg models (X, W) to the case of “logarithmic” Landau-Ginzburg models (X, D, W) , where D is a normal crossings divisor.

We thank our advisor Constantin Teleman for his guidance and support. We would also like to thank Leonid Positselski for his interest in our work and for many useful comments, and Raphael Rouquier for helpful correspondence.

2. CURVED QUASI-COHERENT SHEAVES AND MATRIX FACTORIZATIONS

Let X be a quasi-projective smooth variety over \mathbb{C} and W a regular function such that $X \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is flat. We now consider the dg category of curved complexes of quasi-coherent sheaves $\mathrm{QCoh}(X, W)$, that is, the category with objects

$$E = (E_1 \begin{array}{c} \xrightarrow{e_1} \\ \xleftarrow{e_0} \end{array} E_0)$$

where E_i are quasi-coherent sheaves of \mathcal{O}_X -modules and the e_i are morphisms of \mathcal{O}_X -modules satisfying $e_{i+1} \circ e_i = W \cdot \text{id}_{E_i}$. The morphism complexes are defined exactly as before except with $\text{Hom}_{\mathcal{O}_X}$ rather than Hom_A . We will denote by $E[1]$ the curved complex

$$(E_0 \begin{array}{c} \xrightarrow{-e_0} \\ \xleftarrow{-e_1} \end{array} E_1).$$

Furthermore, one can define the cone of a morphism and a class of exact triangles in $\text{QCoh}(X, W)$ which together with the shift functor $E \mapsto E[1]$ makes the homotopy category $[\text{QCoh}(X, W)] = H^0(\text{QCoh}(X, W))$ a triangulated category. For the details please refer to [Orl1, Orl2]. More generally, given a dg category \mathcal{C} of curved objects, we will let $[\mathcal{C}]$ denote the homotopy category of \mathcal{C} with triangulated category structure defined in the same way. Also, \mathcal{C}_c will denote the full subcategory of \mathcal{C} consisting of objects whose image in the triangulated category $[\mathcal{C}]$ is compact.

Definition 2.1. Denote by $\text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)] \subset [\text{QCoh}(X, W)]$ the thick triangulated subcategory generated by³ the total curved complexes of exact triples of curved quasi-coherent \mathcal{O}_X -modules. Objects of $\text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)]$ are called *acyclic*. The triangulated category $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$ is defined to be the quotient triangulated category $[\text{QCoh}(X, W)] / \text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)]$. We call this category the *absolute derived category*. This definition is also used in [Pos, Orl2].

Remark 2.2. Note that in our curved situation, we are unable to define the derived category in the usual way by inverting quasi-isomorphisms — we cannot speak of cohomology of a *curved* complex, and thus we cannot speak of quasi-isomorphism of curved complexes. Similarly, the usual notion of acyclicity does not make sense.

In the case of ordinary uncurved complexes of sheaves, the total complex of an exact sequence of complexes is acyclic. This motivates the definition of acyclicity and absolute derived category.

Lemma 2.3. *Let H be a triangulated category and A, F be full triangulated subcategories. Then the natural functor $F/(A \cap F) \rightarrow H/A$ is an equivalence of triangulated categories if for any object $X \in H$ there exists an object $Y \in F$ together with a morphism $X \rightarrow Y$ in H such that a cone of that morphism belongs to A .*

Proposition 2.4. *Denote by $\text{Inj}(X, W)$ the full subcategory of $\text{QCoh}(X, W)$ consisting of curved complexes of injective quasi-coherent sheaves. The natural functor $[\text{Inj}(X, W)] \rightarrow \mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$ is an equivalence of triangulated categories. Therefore the category $\text{Inj}(X, W)$ defines a dg enhancement of $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$.*

Proof. This is a scheme theoretic version of Theorems 3.5 and 3.6 of [Pos] and is proved in exactly the same way. We give a sketch of the proof. We wish to apply the previous lemma and so we proceed in two steps.

The first step is very general — we claim that if $B \in \text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)]$ and I is a curved complex of injective sheaves, then $\text{Hom}(B, I)$ is an acyclic complex. Indeed if B is the total curved module of an exact sequence of curved modules

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0,$$

then $\text{Hom}(B, I)$ is the total complex of the exact sequence of complexes

$$0 \rightarrow \text{Hom}(N, I) \rightarrow \text{Hom}(M, I) \rightarrow \text{Hom}(L, I) \rightarrow 0,$$

³This means that we take recursively all shifts, cones, and direct summands [Rou].

so it is acyclic. Since $\text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)]$ is the thick triangulated subcategory generated by such B , the claim follows. We see immediately that $\text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)] \cap \text{Inj}(X, W) = 0$.

It remains to show that for each $B \in \text{QCoh}(X, W)$, there is a morphism $r : B \rightarrow J$ such that $J \in \text{Inj}(X, W)$ and $\text{Cone}(r) \in \text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)]$. Indeed, there is an embedding of any curved complex of quasi-coherent sheaves B into a curved complex of injectives G_0 . To see this, note that the underlying graded sheaf $B^\#$ embeds into an injective graded quasi-coherent sheaf I_0 , as the category of quasi-coherent sheaves has enough injectives. One then takes the curved complex $G^-(I_0)$ of quasi-coherent sheaves cofreely cogenerated by I_0 (see the proof of Theorem 3.6 of [Pos]), and one checks that B embeds into $G_0 := G^-(I_0)$ and that $G^-(I_0)^\#$ is injective. Let H_0 be the cokernel G_0/B , and similarly we construct a curved complex $G_1 = G^-(I_1)$ of injectives into which H_0 embeds. Proceeding inductively, we obtain a resolution $0 \rightarrow B \rightarrow G_0 \rightarrow G_1 \rightarrow \cdots$ of B by curved complexes of injectives. However, since X is smooth, the category $\text{QCoh}(X)$ of quasi-coherent sheaves has finite homological dimension, and hence for some finite n we must have that the underlying graded sheaf $H_n^\#$ of the cokernel $H_n = G_n/G_{n-1}$ is injective. Let J be the total curved module of the exact complex of curved modules

$$G_0 \rightarrow G_1 \rightarrow \cdots \rightarrow G_n \rightarrow H_n.$$

We are finished. \square

Remark 2.5. Dyckerhoff [Dyc] considers a regular local k -algebra R with maximal ideal \mathfrak{m} and residue field $R/\mathfrak{m} = k$. He takes a superpotential $W \in R$ with isolated singularity at the closed point \mathfrak{m} , and he considers the category $\text{MF}^\infty(R, W)$ consisting of curved complexes of *projective* R -modules of arbitrary rank. In [Pos], it is proved that $[\text{MF}^\infty(R, w)]$ and $\mathbf{D}^{\text{abs}}\text{QCoh}(R, W)$ are equivalent as triangulated categories. In our case, we are forced to use curved complexes of *injective* modules because there are not enough projectives in the global situation.

It is possible to prove the following strengthening of the above proposition:

Theorem 2.6. *There is a model category structure [Hov] on $\text{QCoh}(X, W)$ where a morphism is a weak equivalence if its cone is acyclic. It is cofibrant if it is monic and it is fibrant if it is epic and its kernel is a curved complex of sheaves of injectives. Fibrant objects are sheaves of injectives.*

In the language of [Toë], we have $\text{Inj}(X, W) = \text{Int}(\text{QCoh}(X, W))$ where $\text{Int}(-)$ denotes the full subcategory consisting of objects which are both fibrant and cofibrant. The proof of the above theorem is again similar to Positselski's discussion in the affine case and we omit it since we will not need the full strength of the theorem in the rest of this paper.

In view of the theorem, it is interesting to speculate that the functor $\text{Inj}(-, W)$ defines a fibrant sheaf of dg categories on the Zariski site of X in an appropriate model category structure but the current literature on sheaves of dg-categories appears to be insufficient for making a precise conjecture.

Now, given two curved complexes of quasi-coherent sheaves F and F' , we have an uncurved complex of sheaves $\mathcal{H}\text{om}(F, F')$ which is defined by

$$U \mapsto \text{Hom}_{\mathcal{O}_U}(F|_U, F'|_U).$$

Definition 2.7. We may define a derived functor

$$\mathbb{R}\text{Hom} : \mathbf{D}^{\text{abs}}\text{QCoh}(X, W) \times \mathbf{D}^{\text{abs}}\text{QCoh}(X, W) \rightarrow \mathbf{D}\text{Mod}(\mathcal{O}_X).$$

It is defined by first doing at least one of the following:

- (1) replacing the first argument by a weakly equivalent curved complex of locally free sheaves

(2) replacing the second argument by a weakly equivalent curved complex of injectives and then taking $\mathcal{H}\text{om}$. We have another derived functor

$$\mathbb{R}\text{Hom} : \mathbf{D}^{\text{abs}}\text{QCoh}(X, W) \times \mathbf{D}^{\text{abs}}\text{QCoh}(X, W) \rightarrow \mathbf{D}\text{Mod}(\mathbb{C})$$

defined by fibrant replacement in the second argument.

We now define two different categories of matrix factorizations (same objects, but different morphisms).

Definition 2.8. Define $\text{mf}(X, W)$, $\text{Acycl}^{\text{abs}}[\text{mf}(X, W)]$, and $\mathbf{D}^{\text{abs}}\text{mf}(X, W)$ in the same way as we defined the analogous respective QCoh entities above, except here the objects are curved complexes of locally free sheaves of finite type, i.e. curved vector bundles, i.e. matrix factorizations.

Definition 2.9. Denote by $\text{MF}_{\text{dg}}(X, W)$ the full dg subcategory of $\text{Inj}(X, W)$ consisting of objects weakly equivalent to matrix factorizations.

Remark 2.10. What we call $\mathbf{D}^{\text{abs}}\text{mf}(X, W)$ agrees with what Orlov calls $\text{MF}_0(X, W)$ in [Orl2]. Recall from the introduction that we are assuming in this paper that W has only one singular value $0 \in \mathbb{C}$.

In some arguments it will be convenient to use a third definition: a Čech model of $\text{MF}_{\text{dg}}(X, W)$. Let $\mathcal{U} = \{U_i = \text{Spec } A_i\}$ be a finite covering of X by affine subsets. We follow the notation of §III.4 of [Har], and we write $\mathcal{C}^\bullet(\mathcal{U}, F)$ for the sheaf Čech complex of a sheaf F . We define the dg category $\text{MF}_{\text{Cech}}(X, W)$ as follows: the objects are matrix factorizations; the morphisms $\text{Hom}_{\text{MF}_{\text{Cech}}}(P, P')$ are given by the global sections of the total complex of the double complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{H}\text{om}(P, P'))$ with the first differential being the Čech differential and the second differential induced by that of $\mathcal{H}\text{om}(P, P')$. Although $\text{MF}_{\text{Cech}}(X, W)$ depends on the covering \mathcal{U} , we suppress this from the notation because different coverings yield weakly equivalent dg categories⁴. It is a tedious but standard consideration to see the following:

Proposition 2.11. *We have a weak equivalence $\text{MF}_{\text{Cech}}(X, W) \rightarrow \text{MF}_{\text{dg}}(X, W)$ of dg categories.*

Definition 2.12 ([Orl1]). For any quasi-projective scheme Y over \mathbb{C} , we denote by $\mathfrak{Pctf}(Y)$ the thick subcategory of $\mathbf{D}^b\text{Coh}(Y)$ consisting of perfect objects, and we define

$$\mathbf{D}_{\text{Sing}}^b(Y) = \mathbf{D}^b\text{Coh}(Y)/\mathfrak{Pctf}(Y).$$

Proposition 2.13. *$[\text{MF}_{\text{dg}}(X, W)]$ is equivalent to $\mathbf{D}_{\text{Sing}}^b(X_0)$, where X_0 denotes the fiber $W^{-1}(0)$.*

Proof. There is a natural triangulated functor $\text{coker} : [\text{mf}(X, W)] \rightarrow \mathbf{D}_{\text{Sing}}^b(X_0)$ given by

$$P = (P_1 \begin{array}{c} \xrightarrow{p_1} \\ \xleftarrow{p_0} \end{array} P_0) \mapsto \text{coker}(p_1) =: \text{coker}(P).$$

Let $\{U_i\}$ be as above an affine open cover of X . Consider an object P whose image in the homotopy category lies in $\text{Acycl}^{\text{abs}}[\text{mf}(X, W)]$ and restrict it to one of the affine opens U_i . The image of $P|_{U_i}$ is in $\text{Acycl}^{\text{abs}}[\text{mf}(U_i, W)]$. By an argument similar to the first part of the proof of Proposition 2.4 (except in this situation consider projectives instead of injectives), this subcategory is 0, which means that $P|_{U_i}$ is contractible and hence its cokernel is locally free [Orl1]. Since this holds for each U_i we conclude that $\text{coker}(P)$ is locally free and therefore vanishes in $\mathbf{D}_{\text{Sing}}^b(X_0)$. Thus the coker functor factors through

⁴The category of dg categories has a model category structure for which weak equivalences are quasi-isomorphisms of dg categories.

$\mathbf{D}^{\text{abs}}\text{mf}(X, W)$, and [Orl2] proves that the induced functor $\mathbf{D}^{\text{abs}}\text{mf}(X, W) \rightarrow \mathbf{D}_{\text{Sing}}^{\text{b}}(X_0)$ is an equivalence of triangulated categories.

To prove the proposition, it suffices to show that the natural functor $\mathbf{D}^{\text{abs}}\text{mf}(X, W) \rightarrow \mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$ is fully faithful. For this purpose, it is useful to consider the categories $\text{Coh}(X, W)$, $\text{Acycl}^{\text{abs}}[\text{Coh}(X, W)]$, and $\mathbf{D}^{\text{abs}}\text{Coh}(X, W)$ defined in the same way as the respective mf and QCoh entities. By Exercise II.5.15 of [Har], any morphism from a curved coherent sheaf $F \in \text{Coh}(X, W)$ to an acyclic curved quasi-coherent sheaf $A \in \text{Acycl}^{\text{abs}}[\text{QCoh}(X, W)]$ factors through an acyclic curved coherent sheaf $A' \in \text{Acycl}^{\text{abs}}[\text{Coh}(X, W)]$. It follows that $\mathbf{D}^{\text{abs}}\text{Coh}(X, W) \rightarrow \mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$ is fully faithful.

It remains to show that $\mathbf{D}^{\text{abs}}\text{mf}(X, W) \rightarrow \mathbf{D}^{\text{abs}}\text{Coh}(X, W)$ is fully faithful. To see this, note that since we are on a smooth scheme, any coherent sheaf has a finite resolution by vector bundles. Therefore, following a similar argument as in Proposition 2.4, for any curved coherent sheaf C we can produce a triangle $F \rightarrow C \rightarrow A$ where F is a matrix factorization and A is acyclic. It follows from the dual version of Lemma 2.3 that

$$[\text{mf}(X, W)]/([\text{mf}(X, W)] \cap \text{Acycl}^{\text{abs}}[\text{Coh}(X, W)]) \cong \mathbf{D}^{\text{abs}}\text{Coh}(X, W).$$

The thick subcategory $[\text{mf}(X, W)] \cap \text{Acycl}^{\text{abs}}[\text{Coh}(X, W)]$ can be identified with the thick subcategory $\text{Acycl}^{\text{abs}}[\text{mf}(X, W)]$. We see this as follows. Taking $\text{coker}(P)$ of objects P of the former category gives the zero object in $\mathbf{D}_{\text{Sing}}^{\text{b}}(X_0)$ by the same local argument explained at the beginning of this proof. By Orlov's result mentioned above, it follows that P must have been equivalent to an object in $\text{Acycl}^{\text{abs}}[\text{mf}(X, W)]$ to begin with. \square

Remark 2.14. The cokernel of an acyclic curved coherent sheaf does not necessarily represent the zero object in $\mathbf{D}_{\text{Sing}}^{\text{b}}(X_0)$. However, one may still define a derived cokernel functor $\mathbf{D}^{\text{abs}}\text{Coh}(X, W) \rightarrow \mathbf{D}_{\text{Sing}}^{\text{b}}(X_0)$ by composing the functor $\mathbf{D}^{\text{abs}}\text{mf}(X, W) \rightarrow \mathbf{D}_{\text{Sing}}^{\text{b}}(X_0)$ with an inverse to the equivalence $\mathbf{D}^{\text{abs}}\text{mf}(X, W) \rightarrow \mathbf{D}^{\text{abs}}\text{Coh}(X, W)$ from above.

Proposition 2.15. *Objects of $[\text{MF}_{\text{dg}}(X, W)]$ are compact as objects of $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$*

Proof. Let P be a matrix factorization and Q an arbitrary curved quasi-coherent sheaf. It is standard to see that $\mathbb{R}\text{Hom}(P, Q)$ can be computed using the complex $\Gamma \text{Tot } \mathcal{C}^{\bullet}(\mathcal{U}, \text{Hom}(P, Q))$.

Since the U_i (and their intersections U_{ij} , etc.) are affine, it follows that the restrictions $P|_{U_i}$ is compact in $\mathbf{D}^{\text{abs}}\text{QCoh}(U_i, W)$ by [Pos] (and analogously for the intersections U_{ij} , etc.). Let $Q = \bigoplus_i Q_i$ be a direct sum of curved quasi-coherent sheaves. We have that $\Gamma \text{Tot } \mathcal{C}^{\bullet}(\mathcal{U}, \text{Hom}(P, \bigoplus_i Q_i)) \cong \Gamma \text{Tot } \mathcal{C}^{\bullet}(\mathcal{U}, \bigoplus_i \text{Hom}(P, Q_i))$, because the restrictions $P|_{U_i}$, $P|_{U_{ij}}$, etc. are compact, and finally we have $\Gamma \text{Tot } \mathcal{C}^{\bullet}(\mathcal{U}, \bigoplus_i \text{Hom}(P, Q_i)) = \bigoplus_i \Gamma \text{Tot } \mathcal{C}^{\bullet}(\mathcal{U}, \text{Hom}(P, Q_i))$. \square

For what follows, we need the following well-known lemma [BonVDB]:

Lemma 2.16. *Let \mathcal{T} be a triangulated category with arbitrary direct sums and which is compactly generated by a set of objects C . Then the set of compact objects of \mathcal{T} is C^{thk} , the thick closure of C .*

Proposition 2.17. $\overline{[\text{MF}_{\text{dg}}(X, W)]} \cong \mathbf{D}^{\text{abs}}\text{QCoh}(X, W)_c$, where the notation $\overline{\mathcal{C}}$ means idempotent completion of a category \mathcal{C} .

Proof. By the above lemma it suffices to prove that $[\text{MF}_{\text{dg}}(X, W)] \cong \mathbf{D}^{\text{abs}}\text{mf}(X, W) \cong \mathbf{D}^{\text{abs}}\text{Coh}(X, W)$ generates $\mathbf{D}^{\text{abs}}\text{QCoh}(X, W)$. What we want to prove is the global version of Theorem 2 on page 43 of [Pos] and the proof is very similar. Let J be an object of $\text{Inj}(X, W)$, By the standard Bousfield localization

argument, what we have to show is that if $\text{Hom}(B, J)$ is acyclic for every coherent curved module B , then J is contractible, meaning that it is weakly equivalent to the zero object.

Consider the ordered set of pairs (C, h) , where C is a curved quasi-coherent subsheaf of J and h is a contracting homotopy for the inclusion $C \hookrightarrow J$. Using Zorn's lemma, let (M, h) be a maximal such pair. We show that if $M \neq J$, then $M \hookrightarrow J$ factors through some $M' \hookrightarrow J$, and the contracting homotopy h extends to a contracting homotopy h' for $M' \hookrightarrow J$. From here the result follows.

So suppose $M \neq J$. Then again using Exercise II.5.15 of [Har], we can find a curved quasi-coherent subsheaf M' of J such that M' strictly contains M and the quotient M'/M is coherent. Producing the contracting homotopy proceeds exactly as in [Pos]. \square

The following will be used in the next section:

Lemma 2.18. *Let F be a coherent sheaf on $W^{-1}(0) = X_0$ considered as an object of $\text{Coh}(X, W)$. Suppose P is a matrix factorization and $f : P \rightarrow F$ is a morphism of curved sheaves such that $\text{Cone}(f)$ is acyclic. Then $\text{coker}(P) \cong F$ in $\mathbf{D}_{\text{Sing}}^b(X_0)$. (Moreover, such a P exists.)*

Proof. We know that $P \cong F$ in $\mathbf{D}^{\text{abs}}\text{Coh}(X, W)$. First we check that the result holds if F , as a coherent sheaf, is *maximal Cohen–Macaulay*, which means that $\mathcal{E}xt^i(F, \mathcal{O}_{X_0}) = 0$ for $i > 0$. To see this, note that there is a length two resolution of F by locally free sheaves on X (see the proof of Theorem 3.9 in [Orl1])

$$0 \rightarrow Q_1 \rightarrow Q_0 \rightarrow F \rightarrow 0.$$

Let $G^+(Q_0)$ be the free curved module generated by Q_0 (see again Theorem 3.6 of [Pos]). We have a surjection of curved sheaves $G^+(Q_0) \rightarrow F$ whose kernel is isomorphic to $Q[1]$, where

$$Q = (Q_1 \begin{array}{c} \xrightarrow{q_1} \\ \xleftarrow{q_0} \end{array} Q_0),$$

with q_1 the inclusion map and q_0 the homotopy expressing the fact that W kills F .

We clearly have $\text{coker}(Q) = \text{coker}(F) = F$. Since $G^+(Q_0)$ is contractible, we have an isomorphism $Q \cong F$ in $\mathbf{D}^{\text{abs}}\text{Coh}(X, W)$. Hence we have $P \cong F \cong Q$ in $\mathbf{D}^{\text{abs}}\text{Coh}(X, W)$. Previously we checked that the functor $\mathbf{D}^{\text{abs}}\text{mf}(X, W) \rightarrow \mathbf{D}^{\text{abs}}\text{Coh}(X, W)$ is fully faithful, and hence $P \cong Q$ in $\mathbf{D}^{\text{abs}}\text{mf}(X, W)$. Thus $\text{coker}(P) \cong \text{coker}(Q) = F$ in $\mathbf{D}_{\text{Sing}}^b(X_0)$.

For the general case, for any coherent sheaf F , there is a resolution

$$0 \rightarrow F' \rightarrow \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F \rightarrow 0$$

where the F_i are locally free and F' is maximal Cohen–Macaulay. Thus $F \cong F'$ in $\mathbf{D}^{\text{abs}}\text{Coh}(X, W)$ as well as in $\mathbf{D}_{\text{Sing}}^b(X_0)$ and so the lemma is proven. \square

3. COMPACT GENERATORS AND HOCHSCHILD (CO)HOMOLOGY

Let X be as above a smooth quasi-projective variety over \mathbb{C} , and let W be an arbitrary superpotential. The purpose of this section is to prove the following theorem:

Theorem 3.1. *The Hochschild cohomology of the category $\text{MF}_{\text{dg}}(X, W)$ is $\mathbb{R}\Gamma(\Lambda^\bullet T_X, [W, -])$, where $[-, -]$ denotes the Schouten–Nijenhuis bracket.*

Remark 3.2. One can similarly determine that the Hochschild homology is $\mathbb{R}\Gamma(\Omega_X^\bullet, dW \wedge)$. We focus on Hochschild cohomology, both in the interest of brevity and because in the case when X is Calabi–Yau, which is actually our primary case of interest, the Hochschild homology result follows by section 4 of this paper.

The Hochschild cohomology of a dg category can be defined as the derived endofunctors of the identity functor of the category [Toë]. The Hochschild cohomology of $\mathrm{MF}_{\mathrm{dg}}(X, W)$ is the same as that of $\mathrm{Inj}(X, W)$. Consider the category of endofunctors of $\mathrm{Inj}(X, W)$, and consider the full subcategory consisting of continuous functors. We identify this full subcategory with the category $\mathrm{Inj}(X \times X, \widetilde{W})$, where $\widetilde{W} := \pi_1^*(W) - \pi_2^*(W)$. Further, we identify the identity functor with the diagonal curved complex $\Delta \in \mathrm{Inj}(X \times X, \widetilde{W})$. These identifications are all consequences of results regarding compact generators of our categories. This should all be reminiscent of the work of [Dyc] for the case of affine X . The idea of looking at the category of sheaves on the product is of course an old idea (Fourier–Mukai, etc.).

We compute the Hochschild cohomology of $\mathrm{MF}_{\mathrm{dg}}(X, W)$, then, by computing the derived endomorphisms of Δ . The diagonal curved complex Δ is given by, explicitly,

$$(0 \begin{smallmatrix} \xrightarrow{0} \\ \xleftarrow{0} \end{smallmatrix} \mathcal{O}_\Delta),$$

where \mathcal{O}_Δ is the structure sheaf of the diagonal $X \hookrightarrow X \times X$. Observe that \widetilde{W} is identically zero on \mathcal{O}_Δ , and that $\mathrm{coker}(\Delta) = \mathcal{O}_\Delta$.

Lemma 3.3. $\mathbb{R}\mathrm{Hom}(\Delta, \Delta) \cong \mathbb{R}\Gamma(\Lambda^\bullet T_X, [W, -])$.

Proof. We have a functor [PolPos]

$$\mathcal{E}\mathrm{xt}^{\mathrm{II}} : \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \times \mathbf{D}^{\mathrm{abs}}\mathrm{QCoh}(X, W) \rightarrow \mathbf{D}\mathrm{QCoh}(X).$$

which is defined as follows — first do at least one of the following two things:

- (1) replace the first argument with a complex P^\bullet of curved complexes of locally free sheaves
- (2) replace the second argument with a complex I^\bullet of curved complexes of injective sheaves,

then take their $\mathcal{H}\mathrm{om}$, and then finally take the direct sum total complex Tot^\oplus of the resulting triple complex. Because X is smooth and so $\mathrm{QCoh}(X)$ has finite homological dimension we can choose such resolutions to have finite length and thus we have that $\mathcal{E}\mathrm{xt}^{\mathrm{II}}(\Delta, \Delta)$ and $\mathbb{R}\mathcal{H}\mathrm{om}^\bullet(\Delta, \Delta)$ agree.

In this section we follow [Yek], but with a few adaptations to our curved situation. Let \mathfrak{X}^q be the formal completion of $X^q = X \times \cdots \times X$ along the diagonal X . For a commutative algebra A , denote by $B_q(A)$ the q th term $A \otimes A^{\otimes q} \otimes A$ in the standard bar complex $B(A)$. Let $\widehat{\mathcal{B}}_q(A)$ be the I_q -adic completion of $B_q(A)$, where I_q is the kernel of the map $B_q(A) \rightarrow A$ defined by $a_0 \otimes \cdots \otimes a_{q+1} \mapsto a_0 \cdots a_{q+1}$. On $B(A)$ we have the usual bar complex differential ∂_B , and we also have the “curved” differential ∂_W which is defined by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^{i+1} a_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes W \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

Check that $(\partial_B + \partial_W)^2 = (\widetilde{W} \cdot -)$. It is easy to check that ∂_B and ∂_W are continuous with respect to the I -adic topologies.

Define $\widehat{\mathcal{B}}_q(X) := \mathcal{O}_{\mathfrak{X}^{q+2}}$. On an open affine $U = \mathrm{Spec} A \subset X$, we have $\Gamma(U, \widehat{\mathcal{B}}_q(X)) = \widehat{\mathcal{B}}_q(A)$. The ∂_B and ∂_W sheafify to give maps $\widehat{\partial}_B : \widehat{\mathcal{B}}_q(X) \rightarrow \widehat{\mathcal{B}}_{q-1}(X)$ and $\widehat{\partial}_W : \widehat{\mathcal{B}}_q(X) \rightarrow \widehat{\mathcal{B}}_{q+1}(X)$. Now let M be a curved \mathcal{O}_{X^2} -module with curvature \widetilde{W} . We denote the *Hochschild cohomology complex of \mathcal{O}_X with coefficients in M* as $\mathcal{H}\mathrm{och}^\oplus(\mathcal{O}_X, M)$, and it is defined as follows. It is a $\mathbb{Z}/2\mathbb{Z}$ -graded complex with i th component given by

$$\bigoplus_{p+q=i} \mathcal{H}\mathrm{om}_{\mathcal{O}_{X^2}}^{\mathrm{cont}}(\widehat{\mathcal{B}}_q(X), M_p).$$

This complex has differential $\partial + \partial_B + \partial_W$, where ∂ is induced from M , and ∂_B and ∂_W are induced by the respective maps defined above. (See page 24 of [PolPos].) The superscript “cont” denotes continuous morphisms, where we have the adic topology on $\widehat{\mathcal{B}}(X)$ and the discrete topology on M .

The category $\text{Mod}_{\text{disc}} \mathcal{O}_{\mathfrak{X}^2}$ (see §2 of [Yek]) has enough injectives. Consider \mathcal{O}_X as an object of this category. Following the same argument as in Proposition 2.4, the curved module \mathcal{O}_X can be resolved by curved complexes of injective objects in $\text{Mod}_{\text{disc}} \mathcal{O}_{\mathfrak{X}^2}$. Let $0 \rightarrow \mathcal{O}_X \rightarrow I^\bullet$ be such a resolution. Then $0 \rightarrow \mathcal{O}_X \rightarrow I^\bullet$ is also a resolution of \mathcal{O}_X as an \mathcal{O}_{X^2} -module by curved injective \mathcal{O}_{X^2} -modules.

Therefore, we have

$$\mathcal{E}xt_{\mathcal{O}_{X^2}}^{\text{II}}(\mathcal{O}_X, \mathcal{O}_X) = \text{Tot}^\oplus \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, I^\bullet).$$

The authors do not know whether the I^\bullet can be chosen to be quasi-coherent as \mathcal{O}_{X^2} -modules. However, on a locally noetherian scheme Y , since sheaves which are injective as quasi-coherent sheaves are also injective as \mathcal{O}_Y -modules (see [HarRD]), this means that in our situation resolving \mathcal{O}_X by a complex of curved injective but not necessarily quasi-coherent sheaves gives the same $\mathcal{E}xt^{\text{II}}$ result as if we had resolved it by a complex of curved injective quasi-coherent sheaves.

Since all of the sheaves involved are discrete, we have

$$\text{Tot}^\oplus \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, I^\bullet) = \text{Tot}^\oplus \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}^{\text{cont}}(\mathcal{O}_X, I^\bullet).$$

Consider the bicomplex $\mathcal{H}och^\oplus(\mathcal{O}_X, I^\bullet)$ (defined in the obvious way) and the total complex obtained by taking direct sums of the diagonals of this bicomplex. Call this total complex $\mathcal{H}(I^\bullet)$. Then we have two maps to $\mathcal{H}(I^\bullet)$,

$$\text{Tot}^\oplus \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}^{\text{cont}}(\mathcal{O}_X, I^\bullet) \rightarrow \mathcal{H}(I^\bullet) \leftarrow \left(\bigoplus_q \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}^{\text{cont}}(\widehat{\mathcal{B}}_q(X), \mathcal{O}_X), \partial_B + \partial_W \right).$$

The first map is induced by the morphism $\widehat{\mathcal{B}}(X) \rightarrow \mathcal{O}_X$ and is a quasi-isomorphism by a spectral sequence argument. The second map is induced by $\mathcal{O}_X \rightarrow I^\bullet$ and is a quasi-isomorphism by Lemma 2.7 of [Yek], which states that that when X is smooth over \mathbb{C} , the functor

$$\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}^{\text{cont}}(\widehat{\mathcal{B}}_q(X), -) : \text{Mod}_{\text{disc}} \mathcal{O}_{\mathfrak{X}^2} \rightarrow \text{Mod}_{\text{disc}} \mathcal{O}_{\mathfrak{X}^2}$$

is exact. Our argument here parallels the argument on page 25 of [PolPos].

Define $\widehat{\mathcal{C}}(X) := \widehat{\mathcal{B}}(X) \otimes_{\mathcal{O}_{\mathfrak{X}^2}} \mathcal{O}_X$, with induced differential also denoted by $\partial_B + \partial_W$. Then we have an identification of complexes

$$\left(\bigoplus_q \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}^{\text{cont}}(\widehat{\mathcal{B}}_q(X), \mathcal{O}_X), \partial_B + \partial_W \right) \cong \left(\bigoplus_q \mathcal{H}om_{\mathcal{O}_X}^{\text{cont}}(\widehat{\mathcal{C}}_q(X), \mathcal{O}_X), \partial_B + \partial_W \right).$$

We now note that there is a formality quasi-isomorphism

$$\pi : (\Lambda^\bullet T_X, 0) \rightarrow \left(\bigoplus_q \mathcal{H}om_{\mathcal{O}_X}^{\text{cont}}(\widehat{\mathcal{C}}_q(X), \mathcal{O}_X), \partial_B \right).$$

On an affine subscheme $\text{Spec } A$, each graded component of the right hand side can be identified with polydifferential operators, namely the subcomplex of $\text{Hom}_k(A^{\otimes q}, A)$ consisting of maps that are differential operators in each factor, and the isomorphism has the form

$$\pi(v_i \wedge \cdots \wedge v_q)(a_1 \otimes \cdots \otimes a_q) = \frac{1}{q!} \sum_{\sigma \in S_q} \text{sgn}(\sigma) v_{\sigma(1)}(a_1) \cdots v_{\sigma(q)}(a_q).$$

(See pages 13 and 14 of [Yek].) One computes explicitly in these local coordinates that $\pi([W, -]) = \partial_W(\pi(-))$ and thus we get an induced map of complexes

$$\pi : (\Lambda^\bullet T_X, [W, -]) \rightarrow \left(\bigoplus_q \mathcal{H}om_{\mathcal{O}_X}^{\text{cont}}(\widehat{\mathcal{C}}_q, \mathcal{O}_X), \partial_B + \partial_W \right).$$

We conclude using exactly the same spectral sequence argument as in [CalTu] in the affine case that this is a quasi-isomorphism. \square

To complete Theorem 3.1, we need to prove the following, which uses the language of [Toë]:

Theorem 3.4. *We have $\mathbb{R}\text{Hom}_c(\text{Inj}(X_1, Q_1), \text{Inj}(X_2, W_2)) \cong \text{Inj}(X_1 \times X_2, \pi_1^*(W_1) - \pi_2^*(W_2))$. Here $\mathbb{R}\text{Hom}_c$ denotes continuous functors, i.e. functors which commute with arbitrary direct sums. When $X_1 = X_2$ and $W_1 = W_2$, then the induced equivalence of homotopy categories identifies the identity functor with the diagonal curved sheaf Δ as an object of $\mathbf{D}^{\text{abs}}\text{QCoh}(X \times X, \widetilde{W})$.*

We will use the following theorem, which follows by results in section 7 of [Rou].

Theorem 3.5. *If Z is a generator for $\mathbf{D}^b\text{Coh}(\text{Sing}(X))$ and Y is a generator of $\mathfrak{P}\text{erf}(X)$, then $i_*Z \oplus Y$ generates $\mathbf{D}^b\text{Coh}(X)$. Here $\text{Sing}(X)$ denotes the singular locus of X , and i denotes the inclusion $\text{Sing}(X) \hookrightarrow X$.*

It follows that generators of $\mathbf{D}^b\text{Coh}(\text{Sing}(X))$ are also generators of $\mathbf{D}_{\text{Sing}}^b(X)$. We also get a new proof of a result of Dyckerhoff.

Corollary 3.6 ([Dyc]). *If W has exactly one isolated singularity, then the residue field \mathbb{C} of the singularity is a generator of the category $\mathbf{D}_{\text{Sing}}^b(W^{-1}(0)) \cong [\text{MF}_{\text{dg}}(X, W)]$.*

Proof. The structure sheaf is a generator of $\mathbf{D}^b\text{Coh}(\text{Spec } \mathbb{C})$. \square

We will also use the following theorem, which can be proven explicitly for the generators constructed inductively in [Rou], but in the hope that it might be useful in future work, we give a more general statement, the proof of which was outlined to us by Raphael Rouquier.

Theorem 3.7. *If E is a generator of $\mathbf{D}^b\text{Coh}(X)$ and F is a generator of $\mathbf{D}^b\text{Coh}(Y)$ then $E \otimes F$ is a generator of $\mathbf{D}^b\text{Coh}(X \times Y)$.⁵*

Proof. First we observe that if $X = S \cup T$ is the union of two closed subvarieties, and if A generates $\mathbf{D}^b\text{Coh}(S)$ and B generates $\mathbf{D}^b\text{Coh}(T)$, then $A \oplus B$ generates $\mathbf{D}^b\text{Coh}(X)$. We will need to consider the abelian categories of quasi-coherent sheaves with support on S and T , denoted respectively by $\text{QCoh}_S(X)$ and $\text{QCoh}_T(X)$, and their respective derived categories $\mathbf{D}^b\text{QCoh}_S(X)$ and $\mathbf{D}^b\text{QCoh}_T(X)$, as well as the respective Coh entities of coherent sheaves. It follows from Lemma 7.41 of [Rou] that A and B generate $\mathbf{D}^b\text{Coh}_S(X)$ and $\mathbf{D}^b\text{Coh}_T(X)$ respectively and hence also $\mathbf{D}^b\text{QCoh}_S(X)$ and $\mathbf{D}^b\text{QCoh}_T(X)$ respectively. By Proposition 6.15 of [Rou], we know that $\mathbf{D}^b\text{QCoh}(X)_c = \mathbf{D}^b\text{Coh}(X)$ and thus from Lemma 3.13 of [Rou], it is sufficient to prove that $A \oplus B$ generates $\mathbf{D}^b\text{QCoh}(X)$ as a triangulated category with infinite sums. Letting $j : X \setminus S \rightarrow X$ denote the inclusion, it is well-known that if F is an object in $\mathbf{D}^b\text{QCoh}(X)$ then we have a distinguished triangle

$$\mathbb{R}\Gamma_S(F) \rightarrow F \rightarrow j_*j^*F$$

⁵Of course by $E \otimes F$ we mean the external tensor product by abuse of notation.

with $\mathbb{R}\Gamma_S(F)$ in $\mathbf{D}^b\mathrm{QCoh}_S(X)$ and $j_*j^*\mathcal{F}$ in $\mathbf{D}^b\mathrm{QCoh}_T(X)$. This proves the first claim.

Now to prove the theorem, we proceed by induction on $\dim X + \dim Y$. Let E' be a generator of $\mathrm{Sing}(X)$ and F' a generator of $\mathrm{Sing}(Y)$. By induction, we have that $E' \otimes F$ generates $\mathbf{D}^b\mathrm{Coh}(\mathrm{Sing}(X) \times Y)$ and $E \otimes F'$ generates $\mathbf{D}^b\mathrm{Coh}(X \times \mathrm{Sing}(Y))$. Let $Z = (\mathrm{Sing}(X) \times Y) \cup (X \times \mathrm{Sing}(Y))$ which, because we are working over \mathbb{C} , is the same as $\mathrm{Sing}(X \times Y)$. Then $E' \otimes F \oplus E \otimes F'$ generates $\mathbf{D}^b\mathrm{Coh}(Z)$. Let E'' be a generator of $\mathfrak{Pctf}(X)$ and F'' a generator of $\mathfrak{Pctf}(Y)$. Then, $E'' \otimes F''$ generates $\mathfrak{Pctf}(X \times Y)$, hence $(E'' \otimes F'') \oplus (E' \otimes F) \oplus (E \otimes F')$ generates $\mathbf{D}^b\mathrm{Coh}(X \times Y)$. Since each of the three summands is the external tensor product of sheaves, $E \otimes F$ generates $\mathbf{D}^b\mathrm{Coh}(X \times Y)$ as desired. \square

Remark 3.8. As a caution note that the hypothesis that the ground field be \mathbb{C} is important here. The problem is illustrated by the fact that over an imperfect field k , it can happen that X and Y are regular but $X \times Y$ is not. Thus $\mathbf{D}^b\mathrm{Coh}(X)$ and $\mathbf{D}^b\mathrm{Coh}(Y)$ can have perfect generators whose external tensor product will fail to generate $\mathbf{D}^b\mathrm{Coh}(X \times Y)$. As a consequence, the authors don't know of a clean statement for Theorem 3.4 that works over an arbitrary base field.

Dyckerhoff works over an arbitrary field k , but circumvents the above problems by assuming that the residue field R/\mathfrak{m} is also k . Thus for superpotentials with isolated singularities on varieties, the methods described here give an alternative to Dyckerhoff's very explicit construction of a compact generator.

Lemma 3.9. *We have a functor D which takes a matrix factorization P to $\mathcal{H}\mathrm{om}(P, \mathcal{O}_X)$ and which induces an equivalence between $[\mathrm{MF}_{\mathrm{dg}}(X, W)]$ and $[\mathrm{MF}_{\mathrm{dg}}(X, -W)^{\mathrm{op}}]$. We have the following commutative diagram.*

$$\begin{array}{ccccc} \mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X, W) & \longrightarrow & \mathbf{D}^b\mathrm{Coh}(X_0) & \longrightarrow & \mathbf{D}_{\mathrm{Sing}}^b(X_0) \\ \downarrow D & & \downarrow \mathbb{R}\mathrm{Hom}(-, \mathcal{O}_{X_0}[1]) & & \downarrow \mathbb{R}\mathrm{Hom}(-, \mathcal{O}_{X_0}[1]) \\ \mathbf{D}^{\mathrm{abs}}\mathrm{mf}(X, -W)^{\mathrm{op}} & \longrightarrow & \mathbf{D}^b\mathrm{Coh}(X_0)^{\mathrm{op}} & \longrightarrow & \mathbf{D}_{\mathrm{Sing}}^b(X_0)^{\mathrm{op}} \end{array}$$

For a dg category T , we recall the notation $\widehat{T} = \mathrm{Int}(T^{\mathrm{op}}\text{-Mod})$, the full dg subcategory of $T^{\mathrm{op}}\text{-Mod}$ consisting of those T^{op} -modules that are both fibrant and cofibrant [Toë].

Proof of Theorem 3.4. Let E and F be generators of $\mathbf{D}^b\mathrm{Coh}(\mathrm{Sing}(W_1^{-1}(0)))$ and $\mathbf{D}^b\mathrm{Coh}(\mathrm{Sing}(W_2^{-1}(0)))$ respectively. Let P be a matrix factorization of (X_1, W_1) such that we have a triangle $P \rightarrow E \rightarrow C$ with C acyclic, and similarly let Q be a matrix factorization of $(X_2, -W_2)$ such that we have a triangle $Q \rightarrow F \rightarrow C'$ with C' acyclic — we can do this by Lemma 2.18. Let A and B^{op} denote $\mathbb{R}\mathrm{Hom}(P, P)$ and $\mathbb{R}\mathrm{Hom}(Q, Q)$ respectively. Following the same argument as the proof of Theorem 4.2 of [Dyc], we know that $\mathrm{Inj}(X_1, W_1) \cong \widehat{A}$ and we know that $\mathrm{Inj}(X_2, -W_2) \cong \widehat{B^{\mathrm{op}}}$. We also have $\mathrm{Inj}(X_2, W_2) \cong \widehat{B}$.

The cone of $P \otimes Q \rightarrow E \otimes F$ is acyclic. By Theorem 3.7, $E \otimes F$ generates the category $\mathbf{D}_{\mathrm{Sing}}^b(W^{-1}(0))$ where $W = \pi_1^*(W_1) - \pi_2^*(W_2)$, because $\mathrm{Sing}(W^{-1}(0)) = \mathrm{Sing}(W_1^{-1}(0)) \times \mathrm{Sing}(W_2^{-1}(0))$. Therefore by Lemma 2.18, it follows that $P \otimes Q$ generates the matrix factorization category $\mathrm{MF}_{\mathrm{dg}}(X_1 \times X_2, W)$.

Since P and Q are curved vector bundles, we have a canonical isomorphism $\mathcal{H}\mathrm{om}(P, P) \otimes \mathcal{H}\mathrm{om}(Q, Q) \rightarrow \mathcal{H}\mathrm{om}(P \otimes Q, P \otimes Q)$. We then have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(P \otimes Q, P \otimes Q) &= \Gamma \mathrm{Tot} \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{H}\mathrm{om}(P \otimes Q, P \otimes Q)) \\ &\cong \Gamma \mathrm{Tot} \mathcal{C}^\bullet(\mathfrak{U}, \mathcal{H}\mathrm{om}(P, P) \otimes \mathcal{H}\mathrm{om}(Q, Q)) \\ &\cong \mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(P, P) \otimes \mathrm{Hom}_{\mathrm{MF}_{\mathrm{Cech}}}(Q, Q). \end{aligned}$$

Therefore we have

$$\mathrm{Inj}(X_1 \times X_2, W) \cong \widehat{A \otimes B^{\mathrm{op}}}.$$

We conclude with the following string of isomorphisms [Toë]:

$$\mathrm{Inj}(X_1 \times X_2, W) \cong \widehat{A \otimes B^{\mathrm{op}}} \cong \mathbb{R}\mathrm{Hom}_c(\widehat{A}, \widehat{B}) \cong \mathbb{R}\mathrm{Hom}_c(\mathrm{Inj}(X_1, w_1), \mathrm{Inj}(X_2, w_2))$$

In the case of $X_1 = X_2$, the claimed identification of the identity functor with Δ comes from the fact that $\mathbb{R}\mathrm{Hom}(P \otimes D(P), \Delta) \cong A$. \square

Corollary 3.10. *As a corollary of the above calculations and Corollary 1.24 of [Orl1], we conclude that when the critical locus of W is proper, the category $\mathrm{Inj}(X, W)$ is dg affine, proper, and homologically smooth as a differential $\mathbb{Z}/2\mathbb{Z}$ -graded category [KatKonPan].*

Lemma 3.11. *With the same assumptions as the previous corollary, we have the following result*

$$\mathbb{R}\mathrm{Hom}(\mathrm{Inj}(X_1, W_1)_c, \mathrm{Inj}(X_2, W_2)_c) \cong \mathrm{Inj}(X_1 \times X_2, \pi_1^*(W_1) - \pi_2^*(W_2))_c.$$

Proof. Both $\mathrm{Inj}(X_1, W_1)$ and $\mathrm{Inj}(X_2, W_2)$ are equivalent to \widehat{A} and \widehat{B} , where A and B are smooth and proper dg algebras. What we need to know is that if M is an $A \otimes B^{\mathrm{op}}$ -module such that for any perfect A -module P , in particular A itself, $P \otimes M$ is perfect as a B -module, then M is perfect. This follows immediately from the following well-known lemma, see e.g. Proposition 3.4 of [Shk]. \square

Lemma 3.12. *A module N over a smooth and proper dg algebra over k is perfect if and only if $\dim_k H^\bullet(N)$ is finite.*

4. CALABI–YAU PROPERTY

The goal of this section is to prove the following theorem:

Theorem 4.1. *Let (X, W) be as above and, in addition, suppose X is Calabi–Yau. Then the category $\mathrm{Inj}(X, W)_c$ is a Calabi–Yau category of dimension n where n is the dimension of the variety.*

As above, let \widetilde{W} be the function $\pi_1^*(W) - \pi_2^*(W)$ on $X \times X$. Denote $\widetilde{W}^{-1}(0)$ by S . In the previous section we have proved that $\mathrm{Inj}(X, W) \cong \widehat{A}$, where $A = \mathbb{R}\mathrm{Hom}(P, P)$ and P is a compact generator. Let $A^e = A \otimes A^{\mathrm{op}}$ and recall that the inverse Serre bimodule is defined as

$$A^! = \mathbb{R}\mathrm{Hom}_{(A^e)^{\mathrm{op}}}(A, A^e).$$

Thus to prove the Calabi–Yau property it suffices to prove that $A^! \cong A[n]$ in $[\mathrm{Int}(A^e\text{-Mod})]$.

We need to recall some theory from [HarRD]. First we recall that given a closed immersion $i : X \rightarrow Y$ there is a functor

$$i^b = \mathbb{R}\mathrm{Hom}_Y(i_* \mathcal{O}_X, -) : \mathbf{D}^b \mathrm{Coh}(Y) \rightarrow \mathbf{D}^b \mathrm{Coh}(X).$$

It is easy to check that this functor has the property that given two morphisms i and j , we have $(j \circ i)^b \cong i^b \circ j^b$. Now we can factor the diagonal morphism $\Delta : X \rightarrow X \times X$ as the composition of $i : X \rightarrow S$ and $j : S \rightarrow X \times X$, so by the Fundamental Lemma on page 179 of [HarRD],

$$(\Delta)^b(\mathcal{O}_{X \times X}) = \mathbb{R}\mathrm{Hom}_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_{X \times X}) = (\mathcal{O}_\Delta) \otimes \omega_{X/\mathbb{C}}^\vee[n],$$

where $\omega_{X/\mathbb{C}}$ is the canonical sheaf. The right hand side is $\mathcal{O}_\Delta[n]$ when X is Calabi–Yau. A simple calculation shows that $j^b(\mathcal{O}_{X \times X}) = \mathcal{O}_S[1]$. Thus we conclude that

$$\mathbb{R}\mathrm{Hom}_S(\mathcal{O}_\Delta, \mathcal{O}_S[1]) = \mathcal{O}_\Delta[n].$$

From here this argument follows exactly the argument of Lemma 5.9 of [Dyc]. We repeat it here to show how to adapt it to our situation. Consider $D(P) \otimes P$, which is a generator for the category $MF(X \times X, \widetilde{W})$. For any Z , we have

$$\mathbb{R}\mathrm{Hom}(D(P) \otimes P, Z) \cong \mathbb{R}\mathrm{Hom}(D(Z), P \otimes D(P)).$$

Now we let Z be the diagonal shifted by (the parity of) the dimension of X . By the discussion above and Lemma 3.9, $D(Z)$ corresponds to the diagonal Δ . We conclude with the sequence of isomorphisms

$$\begin{aligned} A[n] &\cong \mathbb{R}\mathrm{Hom}(D(P) \otimes P, Z) \\ &\cong \mathbb{R}\mathrm{Hom}(D(Z), P \otimes D(P)) \\ &\cong \mathbb{R}\mathrm{Hom}_{(A^e)^{\mathrm{op}}}(\mathbb{R}\mathrm{Hom}(P \otimes D(P), D(Z)), A^e) \\ &\cong \mathbb{R}\mathrm{Hom}_{(A^e)^{\mathrm{op}}}(A, A^e) \\ &= A^! \end{aligned}$$

as desired.

It would be interesting to make the Calabi–Yau structure more explicit using the theory of residues as has been carried out in the isolated singularities case [DycMur].

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